Lecture 4

Magnetostatics, Boundary Conditions, and Jump Conditions

In the previous lecture, Maxwell's equations become greatly simplified in the static limit. We have looked at how the electrostatic problems are solved. We now look at the magnetostatic case. In addition, we will study boundary conditions and jump conditions at an interface, and how they are derived from Maxwell's equations. Maxwell's equations can be first solved in different domains. Then the solutions are pieced (or sewn) together by imposing boundary conditions at the boundaries or interfaces of the domains. Such problems are called boundary-value problems (BVPs).

4.1 Magnetostatics

From Maxwell's equations, we can deduce that the magnetostatic equations for the magnetic field and flux when $\partial/\partial t = 0$, which are [32, 33, 48]

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{4.1.1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4.1.2}$$

In addition to the above, we have the constitutive relation that $\mathbf{B} = \mu\mathbf{H}$. These two equations are greatly simplified, and hence, are easier to solve compared to the time-varying case. One way to satisfy the second equation is to let

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{4.1.3}$$

because of the vector identity

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \tag{4.1.4}$$

The above is zero for the same algebraic reason that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$. In this manner, Gauss's law in (4.1.2) is automatically satisfied.

From (4.1.1), we have

$$\nabla \times \left(\frac{\mathbf{B}}{\mu} \right) = \mathbf{J} \tag{4.1.5}$$

Then using (4.1.3) into the above,

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{A} \right) = \mathbf{J} \tag{4.1.6}$$

In a homogeneous medium,[1] μ or $1/\mu$ is a constant and it commutes with the differential ∇ operator or that it can be taken outside the differential operator. As such, one arrives at

$$\nabla \times (\nabla \times \mathbf{A}) = \mu \mathbf{J} \tag{4.1.7}$$

We use the vector identity that (see back-of-cab formula in the previous lecture)

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{A}) &= \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A} \\ &= \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \end{aligned} \tag{4.1.8}$$

where ∇^2 is a shorthand notation for $\nabla \cdot \nabla$. As a result, we arrive at [49]

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} \tag{4.1.9}$$

By imposing the Coulomb gauge that $\nabla \cdot \mathbf{A} = 0$, which will be elaborated in the next section, we arrive at the simplified equation

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \tag{4.1.10}$$

The above is also known as the vector Poisson's equation. In cartesian coordinates, the above can be viewed as three scalar Poisson's equations. Each of the Poisson's equation can be solved using the Green's function method previously described. Consequently, in free space

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \iiint_V \frac{\mathbf{J}(\mathbf{r}')}{R} dV' \tag{4.1.11}$$

where

$$R = |\mathbf{r} - \mathbf{r}'| \tag{4.1.12}$$

is the distance between the source point $\mathbf{r}'$ and the observation point $\mathbf{r}$. Here, $dV' = dx'dy'dz'$. It is also variously written as $d\mathbf{r}'$ or $d^3\mathbf{r}'$.

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[1] Its prudent to warn the reader of the use of the word "homogeneous". In the math community, it usually refers to something to be set to zero. But in the electromagnetics community, it refers to something "non-heterogeneous".

4.1.1 More on Coulomb Gauge

Gauge is a very important concept in physics [47], and we will further elaborate it here. First, notice that $\mathbf{A}$ in (4.1.3) is not unique because one can always define

$$\mathbf{A}' = \mathbf{A} - \nabla\Psi \tag{4.1.13}$$

Then

$$\nabla \times \mathbf{A}' = \nabla \times (\mathbf{A} - \nabla\Psi) = \nabla \times \mathbf{A} = \mathbf{B} \tag{4.1.14}$$

where we have made use of that $\nabla \times \nabla\Psi = 0$. Hence, the $\nabla\times$ of both $\mathbf{A}$ and $\mathbf{A}'$ produce the same $\mathbf{B}$; hence, $\mathbf{A}$ is non-unique.

To find $\mathbf{A}$ uniquely, we have to define or set the divergence of $\mathbf{A}$ or provide a gauge condition. One way is to set the divergence of $\mathbf{A}$ to be zero, namely that

$$\nabla \cdot \mathbf{A} = 0 \tag{4.1.15}$$

Then

$$\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} - \nabla^2\Psi \neq \nabla \cdot \mathbf{A} \tag{4.1.16}$$

The last non-equal sign follows if $\nabla^2\Psi \neq 0$. However, if we further stipulate that $\nabla \cdot \mathbf{A}' = \nabla \cdot \mathbf{A} = 0$, then $-\nabla^2\Psi = 0$. This does not necessary imply that $\Psi = 0$, but if we impose that condition that $\Psi \to 0$ when $\mathbf{r} \to \infty$, then $\Psi = 0$ everywhere.[2] By so doing, $\mathbf{A}$ and $\mathbf{A}'$ are equal to each other, and we obtain (4.1.10) and (4.1.11).

The above is akin to the idea that given a vector $\mathbf{a}$, just by stipulating that $\mathbf{b} \times \mathbf{a} = \mathbf{c}$ is not enough to determine $\mathbf{a}$. We need to stipulate what $\mathbf{b} \cdot \mathbf{a}$ is as well. Here, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are independent vectors. Another way of saying this is that the vector $\mathbf{a}$ can be written as $\mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp}$ where $\mathbf{a}_{\parallel}$ is parallel to the vector $\mathbf{b}$, while $\mathbf{a}_{\perp}$ is perpendicular to $\mathbf{b}$. Here, $\mathbf{a}_{\parallel}$ is indeterminate now, since $\mathbf{b} \times \mathbf{a}_{\parallel} = 0$. But by letting $\mathbf{b} \cdot \mathbf{a} = 0$ will force $\mathbf{a}_{\parallel} = 0$.

4.2 Boundary Conditions—1D Poisson's Equation

To simplify the solutions of Maxwell's equations, they are usually solved in a homogeneous medium. As mentioned before, a complex problem can be divided into piecewise homogeneous regions first, and then the solution in each region sought separately. Then the total solution must satisfy boundary conditions at the interface between the piecewise homogeneous regions.

What are these boundary conditions? Boundary conditions are actually embedded in the partial differential equations that the potential or the field satisfy. Two important concepts to keep in mind are:

- Differentiation of a function with discontinuous slope will give rise to step discontinuity.

[2] It is a property of the Laplace boundary value problem that if $\Psi = 0$ on a closed surface S, then $\Psi = 0$ everywhere inside S. Earnshaw's theorem [32] is useful for proving this assertion.

- Differentiation of a function with step discontinuity will give rise to a Dirac delta function. This is also called the jump condition, a term often used by the mathematics community [50].

Take for example a one dimensional Poisson's equation that

$$\frac{d}{dx}\varepsilon(x)\frac{d}{dx}\Phi(x) = -\varrho(x) \tag{4.2.1}$$

where $\varepsilon(x)$ represents material property that has the form given in Figure 4.1. One can actually say something about $\Phi(x)$ given $\varrho(x)$ on the right-hand side. If $\varrho(x)$ has a delta function singularity, it implies that $\varepsilon(x)\frac{d}{dx}\Phi(x)$ has a step discontinuity. If $\varrho(x)$ is finite everywhere, then $\varepsilon(x)\frac{d}{dx}\Phi(x)$ must be continuous everywhere.

Furthermore, if $\varepsilon(x)\frac{d}{dx}\Phi(x)$ is finite everywhere, it implies that $\Phi(x)$ must be continuous everywhere.

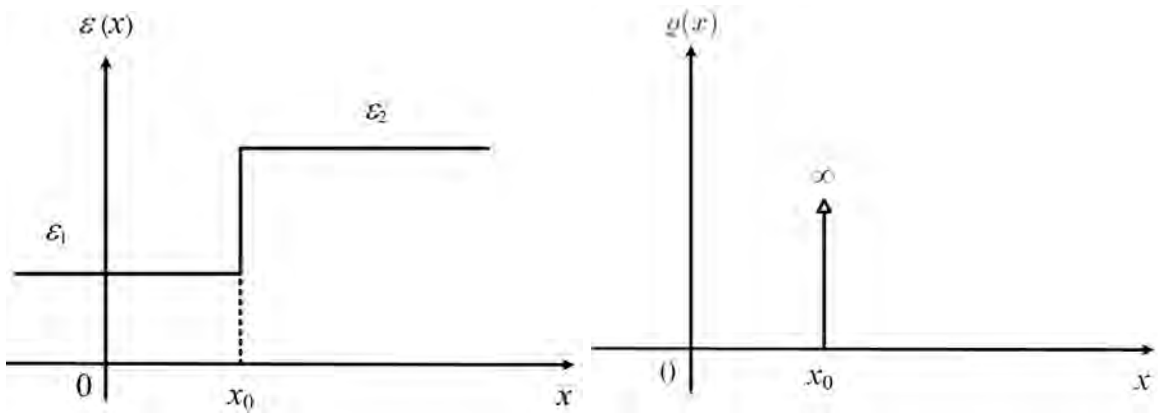

Figure 4.1: A figure showing a charge sheet at the interface between two dielectric media. Because it is a surface charge sheet, the volume charge density $\varrho(x)$ is infinite at the sheet location x_0.

To see this in greater detail, we illustrate it with the following example. In the above, $\varrho(x)$ represents a singular charge distribution given by $\varrho(x) = \varrho_s\delta(x - x_0)$. In this case, the charge distribution is everywhere zero except at the location of the surface charge sheet, where the charge density is infinite: it is represented mathematically by a delta function[3] in space.

To find the boundary condition of the potential $\Phi(x)$ at x_0 , we integrate (4.2.1) over an infinitesimal width around x_0, the location of the charge sheet, namely

$$\int_{x_0-\Delta}^{x_0+\Delta} dx \left[\frac{d}{dx}\varepsilon(x)\frac{d}{dx}\Phi(x)\right] = -\int_{x_0-\Delta}^{x_0+\Delta} dx\varrho(x) = -\int_{x_0-\Delta}^{x_0+\Delta} dx\varrho_s\delta(x - x_0) \tag{4.2.2}$$

[3]This function has been attributed to Dirac who used in pervasively, but Cauchy was aware of such a function.

Since the integrand of the left-hand side is an exact derivative, we get

$$\varepsilon(x)\frac{d}{dx}\Phi(x)\Big|_{x_0-\Delta}^{x_0+\Delta} = -\varrho_s \tag{4.2.3}$$

whereas on the right-hand side, we pick up the contribution from the delta function. Evaluating the left-hand side at their limits, one arrives at

$$\varepsilon(x_0^+)\frac{d}{dx}\Phi(x_0^+) - \varepsilon(x_0^-)\frac{d}{dx}\Phi(x_0^-) \cong -\varrho_s, \tag{4.2.4}$$

where $x_0^\pm = \lim_{\Delta\to 0} x_0 \pm \Delta$. In other words, the jump discontinuity is in $\varepsilon(x)\frac{d}{dx}\Phi(x)$ and the amplitude of the jump discontinuity is proportional to the amplitude of the delta function, ϱ_s.

Since $\mathbf{E} = -\nabla\Phi$, or that

$$E_x(x) = -\frac{d}{dx}\Phi(x), \tag{4.2.5}$$

The above implies the boundary condition that

$$\varepsilon(x_0^+)E_x(x_0^+) - \varepsilon(x_0^-)E_x(x_0^-) = \varrho_s \tag{4.2.6}$$

or

$$D_x(x_0^+) - D_x(x_0^-) = \varrho_s \tag{4.2.7}$$

where

$$D_x(x) = \varepsilon(x)E_x(x) \tag{4.2.8}$$

If $\varrho_s = 0$, then the boundary condition becomes $D_x(x_0^+) = D_x(x_0^-)$.

The lesson learned from above is that boundary condition is obtained by integrating the pertinent differential equation over an infinitesimal small segment. In this mathematical way of looking at the boundary condition, one can also eyeball the differential equation and ascertain the terms that will have the jump discontinuity whose derivatives will yield the delta function on the right-hand side.

4.3 Boundary Conditions—Maxwell's Equations

As seen previously, boundary conditions for a field is embedded in the differential equation that the field satisfies. Hence, boundary conditions can be derived from the differential operator forms of Maxwell's equations. In most textbooks, boundary conditions are obtained by integrating Maxwell's equations over a small pill box [32,33,49]. To derive these boundary conditions, we will take an unconventional view: namely to see what sources can induce jump conditions (or jump discontinuities) on the pertinent fields. Boundary conditions are needed at media interfaces, as well as across current or charge sheets. As shall be shown, each of the Maxwell's equations induces a boundary condition at the interface between two media or two regions separated by surface sources.

4.3.1 Faraday's Law

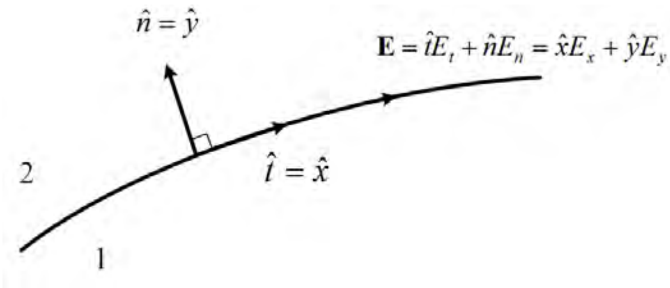

Figure 4.2: This figure is for the derivation of boundary condition induced by Faraday's law. A local coordinate system can be used to see the boundary condition more lucidly. Here, the normal $\hat{n} = \hat{y}$ and the tangential component $\hat{t} = \hat{x}$.

For this, we start with Faraday's law, which implies that

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.3.1}$$

The right-hand side of this equation is a derivative of a time-varying magnetic flux, and it is a finite quantity. One quick answer we could ask is that if the right-hand side of the above equation is everywhere finite, could there be any jump discontinuity on the field $\mathbf{E}$ on the left hand side? The answer is a resounding no!

To see this quickly, one can project the tangential field component and normal field component to a local coordinate system. In other words, one can think of $\hat{t}$ and $\hat{n}$ as the local $\hat{x}$ and $\hat{y}$ coordinates. Then writing the curl operator in this local coordinates, one gets

$$\begin{aligned}\nabla \times \mathbf{E} &= \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right) \times (\hat{x}E_x + \hat{y}E_y) \\ &= \hat{z}\frac{\partial}{\partial x}E_y - \hat{z}\frac{\partial}{\partial y}E_x \end{aligned} \tag{4.3.2}$$

In simplifying the above, we have used the distributive property of cross product, and evaluating the cross product in cartesian coordinates. The cross product gives four terms, but only two of the four terms are non-zero as shown above.

Since the right-hand side of (4.3.1) is finite, the above implies that $\frac{\partial}{\partial x}E_y$ and $\frac{\partial}{\partial y}E_x$ have to be finite. In order words, E_x is continuous in the y direction and E_y is continuous in the x direction. Since in the local coordinate system, $E_x = E_t$, then E_t is continuous across the boundary. The above implies that

$$E_{1t} = E_{2t} \tag{4.3.3}$$

or the tangential components of the electric field is continuous at the interface. To express this in a compact coordinate independent manner, we have

$$\hat{n} \times \mathbf{E}_1 = \hat{n} \times \mathbf{E}_2 \tag{4.3.4}$$

where $\hat{n}$ is the unit normal at the interface, and $\hat{n} \times \mathbf{E}$ always brings out the tangential component of a vector $\mathbf{E}$ (convince yourself).

4.3.2 Gauss's Law for Electric Flux

From this Gauss's law, we have

$$\nabla \cdot \mathbf{D} = \varrho \tag{4.3.5}$$

where ϱ is the volume charge density. We would like to express this equation at an interface in terms of a local self-coordinate system.

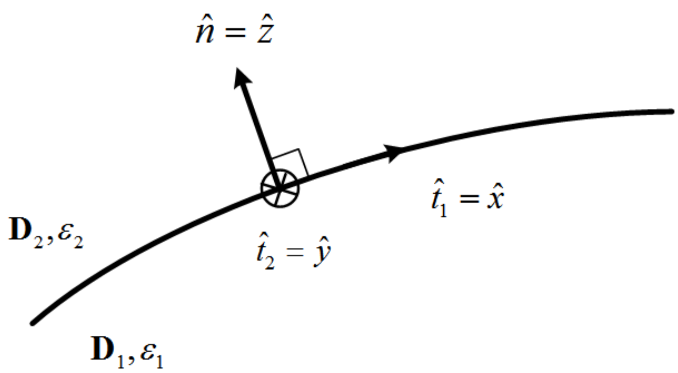

Figure 4.3: A figure showing the derivation of boundary condition for Gauss's law. Again, a local coordinate system can be introduced for simplicity.

Expressing the above in local coordinates (x, y, z as shown in Figure 4.3, then

$$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial x} D_x + \frac{\partial}{\partial y} D_y + \frac{\partial}{\partial z} D_z = \varrho \tag{4.3.6}$$

The boundary condition for the electric flux can be found by *singularity matching.* If there is a surface layer charge at the interface, then the volume charge density must be infinitely large or singular; hence, it can be expressed in terms of a delta function, or $\varrho = \varrho_s \delta(z)$ in local coordinates. Here, ϱ_s is the surface charge density. By looking at the above expression in local coordinates, the only term that can produce a $\delta(z)$ is from $\frac{\partial}{\partial z} D_z$. In other words, D_z has a jump discontinuity at $z = 0$; the other terms do not. Then

$$\frac{\partial}{\partial z} D_z = \varrho_s \delta(z) \tag{4.3.7}$$

Integrating the above from $0-\Delta$ to $0+\Delta$, we get

$$D_z(z)\Big|_{0-\Delta}^{0+\Delta} = \varrho_s \tag{4.3.8}$$

or in the limit when $\Delta \to 0$,

$$D_z(0^+) - D_z(0^-) = \varrho_s \tag{4.3.9}$$

where $0^+ = \lim_{\Delta\to 0} 0 + \Delta$, and $0^- = \lim_{\Delta\to 0} 0 - \Delta$. Since $D_z(0^+) = D_{2n}$, $D_z(0^-) = D_{1n}$, the above becomes

$$D_{2n} - D_{1n} = \varrho_s \tag{4.3.10}$$

In other words, a charge sheet ϱ_s can give rise to a jump discontinuity in the normal component of the electric flux $\mathbf{D}$. Expressed in a compact, coordinate independent form, it is

$$\hat{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \varrho_s \tag{4.3.11}$$

Using the physical notion that an electric charge has electric flux $\mathbf{D}$ exuding from it, Figure 4.4 shows an intuitive sketch as to why a charge sheet gives rise to a discontinuous normal component of the electric flux $\mathbf{D}$.

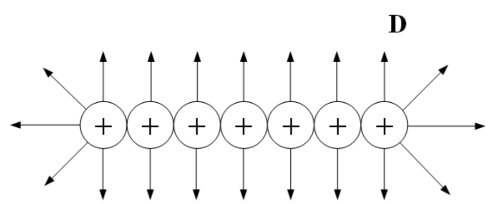

Figure 4.4: A figure intuitively showing why a sheet of charge gives rise to a jump discontinuiy in the normal component of the electric flux $\mathbf{D}$.

4.3.3 Ampere's Law

Ampere's law, or the generalized one, stipulates that

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{4.3.12}$$

Again if the right-hand side is everywhere finite, then $\mathbf{H}$ is a continuous field everywhere. However, if the right-hand side has a delta function singularity, due to a current sheet $\mathbf{J}$ in 3D space, and that $\frac{\partial \mathbf{D}}{\partial t}$ is regular or finite everywhere, then the only place where the singularity can be matched on the left-hand side is from the derivative of the magnetic field $\mathbf{H}$ or $\nabla \times \mathbf{H}$. In a word, $\mathbf{H}$ is not continuous. For instance, we can project the above equation onto a local coordinates just as we did for Faraday's law.

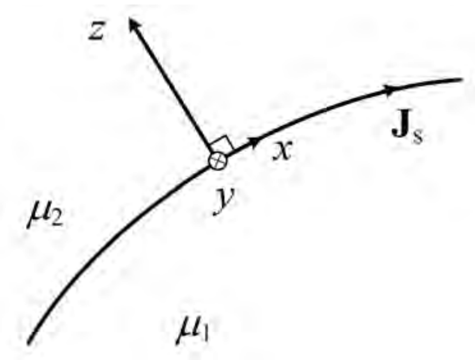

Figure 4.5: A figure showing the derivation of boundary condition for Ampere's law. A local coordinate system is used for simplicity.

To be general, we also include the presence of a current sheet at the interface. A current sheet, or a surface current density becomes a delta function singularity when expressed as a volume current density. Thus, rewriting (4.3.12) in a local coordinate system, assuming that $\mathbf{J} = \hat{x} J_{sx} \delta(z)$,[4] then singularity matching in local coordinates,

$$\nabla \times \mathbf{H} = \hat{x} \left(\frac{\partial}{\partial y} H_z - \frac{\partial}{\partial z} H_y \right) = \hat{x} J_{sx} \delta(z) \tag{4.3.13}$$

The displacement current term on the right-hand side is ignored since it is regular or finite, and will not induce a jump discontinuity on the field; hence, we have the form of the right-hand side of the above equation. From the above, the only term that can produce a $\delta(z)$ singularity on the left-hand side is the $-\frac{\partial}{\partial z} H_y$ term. Therefore, by singularity matching, we conclude that

$$-\frac{\partial}{\partial z} H_y = J_{sx} \delta(z) \tag{4.3.14}$$

In other words, H_y has to have a jump discontinuity at the interface where the current sheet resides; or that

$$H_y(z = 0^+) - H_y(z = 0^-) = -J_{sx} \tag{4.3.15}$$

The above implies that

$$H_{2y} - H_{1y} = -J_{sx} \tag{4.3.16}$$

But H_y is just the tangential component of the $\mathbf{H}$ field. In a word, the current sheet J_{sx} induces a jump discontinuity on the y component of the magnetic field. Now if we repeat the same exercise with a current with a y component, or $\mathbf{J} = \hat{y} J_{sy} \delta(z)$, at the interface, we have

$$H_{2x} - H_{1x} = J_{sy} \tag{4.3.17}$$

[4]The form of this equation can be checked by dimensional analysis. Here, $\mathbf{J}$ has the unit of A m^{-2}, $\delta(z)$ has unit of m^{-1}, and J_{sx}, a current sheet density, has unit of A m^{-1}.

Now, (4.3.16) and (4.3.17) can be rewritten using a cross product as

$$\hat{z} \times (\hat{y}H_{2y} - \hat{y}H_{1y}) = \hat{x}J_{sx} \tag{4.3.18}$$

$$\hat{z} \times (\hat{x}H_{2x} - \hat{x}H_{1x}) = \hat{y}J_{sy} \tag{4.3.19}$$

The above two equations can be combined as one, written in a coordinate independent form, to give

$$\hat{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_s \tag{4.3.20}$$

where in this case here, $\hat{n} = \hat{z}$. In other words, a current sheet $\mathbf{J}_s$ can give rise to a jump discontinuity in the tangential components of the magnetic field, $\hat{n} \times \mathbf{H}$. This is illustrated intuitively in Figure 4.6.

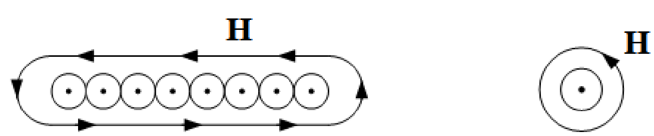

Figure 4.6: A figure intuitively showing that with the understanding of how a single line current source generates a magnetic field (right), a cluster of them forming a current sheet will generate a jump discontinuity in the tangential component of the magnetic field **H** (left).

4.3.4 Gauss's Law for Magnetic Flux

Similarly, from Gauss's law for magnetic flux, or that

$$\nabla \cdot \mathbf{B} = 0 \tag{4.3.21}$$

one deduces that

$$\hat{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = 0 \tag{4.3.22}$$

or that the normal magnetic fluxes are continuous at an interface. In other words, since magnetic charges do not exist, the normal component of the magnetic flux has to be continuous.

The take-home message here is that the boundary conditions are buried in the differential operators and source singularities. If there are singular terms such as sheet sources in Maxwell's equations, then via the differential operators, the boundary conditions can be deduced. These boundary conditions at an interface are also known as jump condition if a current or a source sheet is present.